MATH 3060 Assignment 2 solution

Chan Ki Fung

October 8, 2021

1.

(a) Let the left and right derivatives of f at x be A and B respectively. Then by definition there exists $\delta > 0$ such that

$$\left|\frac{f(y)-f(x)}{y-x} - A\right| < 1$$

whenever

$$x - \delta < y < x,$$

and

$$\left|\frac{f(y)-f(x)}{y-x} - B\right| < 1$$

whenever

$$x < y < x + \delta.$$

Taking $L = \max\{|A|, |B|\} + 1$, we must have

$$|f(y) - f(x)| \leq L|y - x|$$

for

$$|y - x| < \delta$$

(b) A counter example is the function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, x = 0 \\ x \sin \frac{1}{x}, x \neq 0 \end{cases}$$

f is Lipschitz continuous at 0 (with $L = 1$), but neither its left nor right derivative exists at 0.

2. For $0 \le r < 1$.

$$\begin{aligned}
&a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx) \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{k=1}^{\infty} r^k (\int_{-\pi}^{\pi} f(y) \cos ky \cos kx + \sin ky \sin kx dy) \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \sum_{k=1}^{\infty} r^k \cos k(x-y) dy \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)dy + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y) \sum_{k=1}^{\infty} r^k \cos ky dy \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y) \left(\sum_{-\infty}^{\infty} r^{|k|} e^{iky}\right) dy,
\end{aligned}$$

where in the second last equality, we adopt the change of variable $t \mapsto x+y$ and make use of 2π periodicty.

$$\begin{aligned}
\sum_{-\infty}^{\infty} r^{|k|} e^{iky} &= \lim_{N\to\infty} \sum_{k=-N}^{N} r^{|k|} e^{iky} \\
&= \lim_{N\to\infty} \left(\frac{1 - r^{N+1} e^{i(N+1)y}}{1 - re^{iy}} + \frac{1 - r^{N+1} e^{-i(N+1)y}}{1 - re^{-iy}} - 1 \right) \\
&= \frac{1}{1 - re^{iy}} + \frac{1}{1 - re^{-iy}} - 1 \\
&= \frac{1 - r^2}{1 - 2r\cos y + r^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx) \\
=&\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos y + r^2} f(x+y) dy.
\end{aligned}$$

3. For $d \in \mathbb{N}$, let S_d be the set of polynomials of degree less than d and with rational coefficients. Since there are only finite many coefficients, S_d is countable. Now let $S = \cup_{d=1}^{\infty} S_d$ be the set of polynomials (of any degree) with rational coeffcients, then S is also countable. We now claim that S is the required subset.
   Let $f \in C[a, b]$ and $\epsilon > 0$, we know by weierstrass approximation theorem that there exists a polynomial $g' = c_0' + c_1' x + \cdots + c_n' x^n$ such that

$$||f - g'||_\infty < \frac{\epsilon}{2}.$$

Now choose rational numbers $c_0, c_1, \dots c_n$ such that $|c_i - c_i'| \max |a^i|, |b^i| < \frac{\epsilon}{2(n+1)}$. Define $g = c_0 + c_1 x + \cdots + c_n x^n$, then $g \in S$, and

$$\begin{aligned}||f-g||_\infty &\leq ||f-g'||_\infty + ||g'-g||_\infty \\ &< \frac{\epsilon}{2} + \sum_{i=0}^{n} ||(c_i - c_i')x^i|| \\ &< \frac{\epsilon}{2} + \sum_{i=0}^{n} \frac{\epsilon}{2(n+1)} \\ &= \epsilon.\end{aligned}$$

4. Note that $x^3 - \pi x$ is an odd function, so $a_n = 0$ for $n \geq 0$.

$$\begin{aligned}b_n &= \frac{2}{\pi}\int_0^\pi (x^3 - \pi^2 x)\sin nx dx \\ &= \frac{2}{\pi}\left[-\frac{(x^3-\pi^2 x)\cos nx}{n} + \frac{(3x^2-\pi^2)\sin nx}{n^2} + \frac{6x\cos nx}{n^3} - \frac{6\sin nx}{n^4}\right]_0^\pi \\ &= (-1)^n \frac{12}{n^3}.\end{aligned}$$

Therefore,

$$x^3 - \pi^2 x \sim \sum_{n=1}^{\infty} (-1)^n \frac{12}{n^3} \sin nx.$$

On the other hand,

$$\begin{aligned}&\int_{-\pi}^{\pi} (x^3 - \pi^2 x)^2 dx \\ =&\int_{-\pi}^{\pi} x^6 - 2\pi^2 x^4 + \pi^4 x^2 dx \\ =&\left[\frac{x^7}{7} - \frac{2\pi^2 x^5}{5} + \frac{\pi^4 x^3}{3}\right]_{-\pi}^{\pi} \\ =&\frac{16\pi^7}{105}\end{aligned}$$

Therefore, by Parseval's identity,

$$\begin{aligned}\pi \sum_{n=1}^{\infty} \frac{12^2}{n^6} &= \frac{16\pi^7}{105} \\ \sum_{n=1}^{\infty} \frac{1}{n^6} &= \frac{\pi^6}{945}\end{aligned}$$